two-place table of sines and cosines by drawing the angles and measuring the lengths. The graph of the sine curve, and cosine also, is drawn, using the data just obtained. This establishes at once the connection between sines and cosines and waves.

The construction of an angle when one of the ratios is given is a useful application of the graphical method. The angle is measured by means of the protractor, and the other ratios are scaled off. Squared paper is very helpful for this work. Finally, it should be pointed out that at this time the inverse trigonometric functions should be introduced as nomenclature for the angles just constructed, and not, as is usually done, at the very end of the course.

It occurs to the writer that some teachers might contend that in a short course in plane trigonometry there is no time to give to the considerations discussed in this paper. In reply the writer can only cite his experience of this summer. The course covered twenty-eight lecture periods, each fifty minutes in length. Out of these, four were devoted to hour quizzes, and two more, to review at the end of the course. In addition to the graphical introduction, all the topics of the average text on plane trigonometry were covered, including proofs for all the formulæ used, trigonometric identities, and trigonometric equations. The only curtailment necessary was in the time ordinarily devoted to the last topic.

From the point of view of interest it may be stated that this introduction makes a strong appeal to all classes: the future engineer always on the alert for practical methods, the bright student studying mathematics for its own sake, as well as the "prerequisite for the A.B." individual. The method, moreover, is valuable *per se* inasmuch as it is the one used during the late war to solve problems in aerial navigation, artillery and orientation, and plane sailing where first approximations were desired. These fields alone provide a wealth of simple problems.

RECENT PUBLICATIONS.

REVIEWS.

MATHEMATICAL ESSAYS AND ADDRESSES.

The Human Worth of Rigorous Thinking: Essays and Addresses. By CASSIUS J. KEYSER. New York, Columbia University Press, 1916. 314 pp. Price $1.75.

Whitehead and Russell's imposing *Principia Mathematica* furnishes a systematic treatise of the most recent and horoughgoing philosophy of mathematics and as such is addressed to those well grounded in mathematics and logic; Shaw's smaller book, *The Philosophy of Mathematics,* presents the various aspects of mathematics to graduate students. A pleasant task has been assigned to the reviewer, that of describing and evaluating Professor Keyser's delightful essays

and addresses in which he interprets to what are largely non-mathematical audiences the inner spirit of mathematics. Only a mathematician can appreciate the difficulty of his undertaking, and only one who reads or listens to these may know the pleasantness and the effectiveness of the interpretation. These essays are in part known to many through their appearance in such widespread journals as *Science, The Hibbert Journal,* and *The Journal of Philosophy, Psychology and Scientific Methods.*

The style in which these essays are rendered must be at once the satisfaction and the despair of all mathematicians—the satisfaction because the subject has so able an exponent, the despair because so very few indeed—let us say one-half of one per cent.!—of the profession have had their tongues touched by so live a coal from the altar.

Strictly speaking, the title applies only to the first three addresses: "The human worth of rigorous thinking," "The human significance of mathematics," "The humanization of the teaching of mathematics," and to the closing address "Mathematics." Pointing out in his first address that mathematics may be characterized by its aim of thinking rigorously whatever is, or may become, rigorously thinkable, or by the collected results of this rigorous thinking, he examines "the just claims to human regard of perfect thought and the spirit of perfect thinking." To comprehend this fully, one must have come to know what rigorous thinking is through a close study of some one or more distinctively mathematical treatises, of the rôle of rigorous thinking in the history of civilization, and of the numberless applications of mathematics to engineering and the natural sciences. This last study makes it evident that "all thinkers and especially students of natural science are engaged, both consciously and unconsciously, both intentionally and unintentionally, in the mathematization of concepts—that is to say, in so transforming and refining concepts as to fit them finally for the amenities of logic and the austerities of rigorous thinking." As examples of this are adduced the concepts of continuity, function, and infinity. At this phase of the development of the subject he takes issue with Bergson and William James in their attack on the method of concepts, the modern method of science, contending that instead of the aim and ideal of intellect being the understanding and subjugation of matter, the essential function of concepts is something more than this, viz., to quote Diotima, "I am persuaded that all men do all things, and the better they are, the better they do them, in the hope of the glorious fame of immortal virtue." He instances further the great difficulty or utter impossibility which these men have on psychological grounds of explaining such contradictions as those, for example, involved in bridging over the transition from sensations, finite in number, to concepts, infinite in multiplicity. What the intellect has done is rather to create for itself a world of conceptual magnitudes which is free from the contradictions of the world of the senses; these magnitudes form the subject-matter of science. The aim of this creative intellect is to preserve and to promote the life of the intellect itself, to think in accordance with the laws of its being, to bring all its laws and methods into complete harmony. Thus "science

and especially mathematics, the ideal form of science, are creations of the intellect in its quest of harmony."

The second address, delivered at the University of California in 1915 before the American Association for the Advancement of Science and affiliated organizations, is too well known to demand a full analysis. Various approaches to the study of the human significance of mathematics are suggested, the historical, the utilitarian, the logical, and the spiritual, the last furnishing the theme for the address, viz., the relation of mathematics to the supreme ideals of mankind. The sovereign passion of mankind is for release from life's limitations and the tyranny of change. Our human aspirations find their highest unity in the quest of a stable world. This quest is to be made, not in the world of sense, but in that of concepts. When on this road toward perfection thought has attained a high degree of refinement, precision, and rigor, we call it mathematical thought. Yet all thought, mathematical or non-mathematical, refined or crude, possesses the unity of being human. Along with the great contributions of theology, philosophy, jurisprudence, art and science to the wealth of the world's knowledge, mathematics has shown to the world that "there exists a stable world of pure thought, a divinely ordered world of ideas, accessible to man, free from the mad dance of time, infinite and eternal." At the risk of evoking a disclaimer from Professor Keyser, one may fairly say that in his protest against the limitation of the motivation of the intellect to a superiority over the material he shares much in common with the ideals of the pragmatists,[1] while his theory of science, and of mathematics in particular, is almost identical with that of Poincaré.

The thesis of the third address, more briefly summarized, is that hope of improvement in the teaching of mathematics, in secondary schools and colleges alike, lies in the possibility of humanizing it, in recognizing that it is not sufficient to say that mathematics possesses its high position as a great human enterprise because it has given the world a metrical and computatory art essential to the conduct of daily life, furnishes countless applications to engineering and the natural sciences, and is an excellent means of giving mental discipline. It is not sufficient even to say with the mathematician that mathematics is the science of pure thought. Rather is it to be said and to be shown in our teaching that its human significance penetrates all fields, it lives in all the activities of men and of nature.

The next four articles, the thirteenth, and the last may be classed together, treating as they do such general mathematical notions as the dimensions of the universe, hyperspace and infinity. Advisedly avoiding the philosophical questions as to the nature of space and accepting it merely as "a vast region or room around us, an immense eternity, locus of all suspended or floating objects of outer sense," the speaker in "The walls of the world" attempts a bold task in seeking to interpret even to an audience of scientists Pascal's characterization of space: "Space is an infinite sphere whose center is everywhere and whose surface is nowhere." A grasp of the notions of infinite sequences and of the equality of

[1] R. B. Perry, *Present Philosophical Tendencies*, third impression, p. 268.

two such sequences, such as would be sufficient for the comprehension of Pascal's statement, is doubtless not to be gained from a single hearing; fortunate is it that the address is available in a form which will allow those interested to reflect more carefully on the notions involved. While Lucretius's argument for the infinite extent of the universe is in a clear fashion shown to depend on a confusion of the terms "boundless" and "infinite," the suggestion is made through a reference to infinities of higher order that our reason may enable us to go in thought beyond the confines even of our infinite universe.

The next article, "Mathematical emancipations" (1906), and the thirteenth, "Concerning multiple interpretations of postulate systems and the 'existence' of hyperspace" (1913), offer a more scientific treatment of the same subject as the foregoing. The former defines "dimension" with due accuracy and develops instances of magnitudes of successively increasing dimensionality, leading up to the fact that ordinary space is not inherently and uniquely three-dimensional. For example, the plane is a three-dimensional space of circles since each circle is determined by three parameters; and on the other hand ordinary space is by the same test a four-dimensional space of lines or of spheres. This interpretation of hyperspace, which is of course essentially that familiar to mathematicians, has the merit of being simple, of making it unnecessary to pass beyond the bounds of the universe or to transcend the limits of intuition. Hyperspace of points however exists not for imagination or intuition, but only for thought and reason. The question of imagining configurations in a hyperspace of points is a question of psychology, not of mathematics, but it can be readily shown that the structure of a fourfold figure can be traced out by its significant analogy with a three-dimensional figure.[1] On the score of accuracy it should be remarked that "Jove nods" when color is named as an example of three-dimensionality in its dependence on the three primary colors; this is true only in the sense of homogeneous coördinates, for a given color depends manifestly only on the two independent ratios of these three components.

The thirteenth article considers the comparative advantages of the languages of geometry and analysis, and argues that n-dimensional geometry has every kind of existence that may be attributed to ordinary geometry of space.

To the assumptions of modern science that the universe is an organic system of order and that it is the sole system of law and order, "The universe and beyond" opposes the view that above every nature there is a supernatural, a hypercosmic. In support of this view it is noted that while mathematics may be defined as

[1] In the presentation of this subject the reviewer has commonly advanced from the straight line to the square, and from this to the cube, pointing out that in our picture of the cube right angles and lines of equal length do not always appear as such; it is then noted that if a hypercube of four dimensions is to be constructed, one would erect a cube on each of the six faces of this cube extending into the fourth dimension, these cubes being "adjacent" in sets of three, and the "outer vertices," eight in number, would determine the last of the eight cubes which bound the hypercube; the number of edges, faces (squares), and vertices are enumerated by analogy with two and three dimensions, and finally *the picture of the hypercube* (the projection into three-dimensional space) *is given*, this comprising a cube entirely outside a second cube with the pairs of corresponding vertices joined by eight lines.

the science of measurement, direct or indirect, and of position, as the science of operations, etc., as characterized by its content, we may also define it by its method as in the well-known definition of Benjamin Peirce; and since pure mathematics is intrinsically concerned with accuracy, correctness and completeness in its logic rather than with the truth of its applications to our universe, which is the domain of natural science, it may conceivably deal with that which transcends the sensible universe, as when for example one deals in euclidean and non-euclidean geometries with inconsistent systems of consistent relationships.

The last article, "Mathematics," supplements the characterization of mathematics as given in the preceding paper by a presentation of the growth of mathematics as a science through the critical movement grounded in the study of the theory of functions and of the foundations and in a study of symbolic logic which discloses the basis of logic as identical with the basis of mathematics. The background of Keyser's writing seems clearly to be the philosophy of mathematics embodied in Whitehead and Russell's *Principia Mathematica,* the first two volumes of which he has reviewed in the twelfth chapter of the present collection. (The reader who desires to follow the discussion intensively needs to refer for Russell's latest interpretations to his recently published *Introduction to Mathematical Philosophy.*) Hamilton's and Schopenhauer's indictments of mathematics are rejected by Keyser as consisting of malicious misrepresentation; those of Huxley as based on actual ignorance of the nature of mathematical thinking in its relation to observation and causation. It is not surprising that a writer so zealous for the purity and for the spiritual values of mathematics should in the closing part of this paper disparage the term "applied mathematics" and distinguish sharply between mathematics and natural science. This indicates a divergence, partly of emphasis, partly of definition, from the usage of the many who view mathematics as inclusive of its applications.

Incidentally taking issue with the views of Professor Royce as to the possibility and validity of existence proofs of the infinite, the seventh article considers "The axiom of infinity." Mankind by the need of considering the indefinitely small and the indefinitely large has been forced through mathematics, as through ethics and philosophy, to treat the infinite. This search has brought high rewards, even if it were found to be an insoluble problem. The studies of Riemann, Bolzano, Dedekind and Cantor are suggested as having for the first time in history obtained a logical hold on infinity. The contention is rightly made that when some modern critics characterize Bolzano's definition of infinity based on the inexhaustibility of a class as a negative definition, or of the infinite as the really positive and the finite as the negative, the distinction is not essential but represents merely an arbitrarily chosen point of view. Yet, to the reviewer's mind, Professor Keyser is guilty of this very "verbal legerdemain" when he insists that Royce, Russell and others are wrong in saying that the concept of the infinite denies the axiom of the whole and part because the common-sense axiom is applicable only when there is a number telling now many; this contention stands or falls solely with the particular definition given to the terms involved.

The most interesting portion of this essay is the argument[1] that it is impossible to prove the existence of the infinite, that the demonstrations of such existence involve reasoning in a circle, and that one must instead adopt the axiom of infinity which states that "conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly, or, what is the same thing, that the assemblage of possible repetitions of a once performable act is equivalent to some proper part of the assemblage."

The remaining articles are contributions to educational discussions. "The permanent basis of a liberal education" points out that amid the unceasingly changing state of knowledge there are certain great invariant facts which should be enshrined in any plan providing a liberal education; such facts are the immense past of mankind, the material universe in which we find ourselves, our dependence on the world of ideas, the social character of our world, the fact of the human future, and the discipline of beauty.

The address on "Graduate mathematical instruction . . ." calls upon university departments to offer such courses as will enable students to gain a general knowledge of the problems and methods of these fields even though specializing in other departments. It is maintained on the ground of experience and observation that there is an opportunity for such courses and a distinct duty of the universities in the matter, in spite of the fact that the intrinsic technicality of each subject may make it difficult and in part impossible for the instructor to make it known to the non-specialist. The reviewer may in this connection instance the philosopher Josiah Royce, who in the later part of his life with the wealth of his intellectual equipment studied the fundamentals of modern mathematics with the double effect of showing in his writings an unusually intelligent and an unusually sympathetic grasp of mathematics in its philosophical bearings. How happy a contribution might be made to a clean-cut organizing and harmonizing of the various systems of philosophy if only more frequently mathematicians would bring to that study the instrumentality of their critico-logical training! Keyser would demand as a prerequisite only a year of collegiate mathematics, since in his view the material most valuable for such avocational instruction does not depend on the calculus. The course would comprise a presentation of the nature of mathematics, of postulate systems and the rôle they have played in ancient and modern times, the function concept, the limit concept, the calculus, the theory of point sets, invariants, and groups.

Three other chapters are of more general import, one of which, "Mathematical productivity in the United States" (1902) might wisely have been supplemented by the encouraging contrast which a summary of present conditions would furnish.

W. D. Cairns.

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[1] C. J. Keyser, "The axiom of infinity and mathematical induction," *Bulletin of the American Mathematical Society*, Vol. 9, May, 1903.